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Comparison of unitary transforms

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Abstract

We analyse the problem of comparing unitary transformations. The task is to decide, with minimal resources and maximal reliability, whether two given unitary transformations are identical or different. It is possible to make such comparisons without obtaining any information about the individual transformations. Different comparison strategies are presented and compared with respect to their efficiency. With an interferometric setup, it is possible to compare two unitary transforms using only one test particle. Another strategy makes use of a two-particle singlet state. This strategy is more efficient than using a non-entangled two-particle test state, thus demonstrating the benefit of entanglement. Generalizations to higher-dimensional transforms and to more than two transformations are made.

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1. Introduction

Various tasks of quantum information processing require the use of quantum networks which realize desired quantum operations on a chosen input state. The networks may be formed by arrays of elementary gates, realizing single- or two-particle operations. With increasing complexity, it will become important to locate a possible error that can cause the malfunctioning of the whole network. The error, or erroneous gate, could be found by measuring its properties using some estimation strategy [1–11]. The estimation of a whole set of gates, however, might be rather time consuming. Moreover, we may not be interested in what the error is, only if there is an error or not. Therefore, it will be better to perform a simpler check only to see whether the gate performs correctly or not. This leads us to consider testing whether two gates are the same or different by *comparing* them to each other. Similarly it might be favourable to test gates on a production line when a master gate is available. The produced gates could then simply be compared with the master gate, and if a deviation in the behaviour of the gate is found, it will be discarded. Transformation comparison is closely related to state comparison [12].

In the following, we will discuss strategies for comparing unitary transforms. The choice of comparison strategy will naturally depend on what is already known about the transforms, and also on the resources available for the test. In the present work, a restriction is made to when each transform may be used only once, and to when the transforms are completely unknown. Other situations will be commented upon at the end. The focus will also be on unambiguous transformation comparison, meaning that whenever a result ‘different’ or ‘same’ is obtained, the result is also correct. In general, this forces us to accept inconclusive comparison outcomes, giving no information [13]. It is also possible to compare unitary transformations using other types of strategies, such as minimum-error comparison strategies. We will return to this possibility at the end.

It seems reasonable to assume that any unambiguous, minimum-error, or any other type of comparison strategy, would have to involve the action of the transforms to be compared upon some test state, followed by a measurement (or measurements) of the changed test state. To be more specific, we have to build a network where each transform occurs once, feed a test quantum state through the network, and then measure the output state. The transforms may be used in any way inside the network; in parallel, acting on different qubits or particles; in series, acting on a particle one transform after the other, or possibly even in the form of controlled- U operations. This picture includes the possibility of entanglement with an environment, for example, the case where two transforms U and V act in some way on two particles, which may be entangled with other particles (or with each other). To be general, one should also allow for sequential measurements, with actions conditional on previous measurement results. As we allow each transform to be used only once, sequential measurements will be of limited use.

To be able to say anything definite and unambiguous about whether the transforms are the same or different, we have to choose the test state and the network in such a way that certain output results are *unambiguously* associated with the transforms being the same or different. From this point of view, it is clear that, if the unitary transforms are completely unknown, a comparison can never give the definite answer that the transforms (two or more) are identical. To obtain this information, we would have to reliably distinguish between the case when the transforms are exactly the same, and when they are arbitrarily close, but still not identical. That this is not possible can be seen by a continuity argument similar to the one for state comparison [12]. The transforms being the same would have to be associated with the output state lying in a certain known subspace H_{id} of the total allowed state space. When the transforms are almost, but not completely identical, the output state $|\Psi_{\text{diff}}\rangle$ will lie almost, but not completely, in this subspace:

$$|\Psi_{\text{diff}}\rangle = \alpha|\Psi_{\text{id}}\rangle + \beta|\Psi^{\perp}\rangle \quad (1)$$

where $|\Psi_{\text{id}}\rangle$ lies in the subspace H_{id} , and $|\Psi^{\perp}\rangle$ lies in the complementary subspace. By making the transformations arbitrarily close, β can be made arbitrarily small, and there is no measurement that would distinguish unambiguously between $|\Psi_{\text{id}}\rangle$ and $|\Psi_{\text{diff}}\rangle$ for any arbitrarily small β . Such a measurement would have to contain a measurement operator Π_{id} for which, with a finite success probability p_{same} to detect the transforms as ‘same’,

$$\langle\Psi_{\text{id}}|\Pi_{\text{id}}|\Psi_{\text{id}}\rangle = p_{\text{same}} \quad \text{and} \quad \langle\Psi_{\text{diff}}|\Pi_{\text{id}}|\Psi_{\text{diff}}\rangle = 0 \quad \forall\beta. \quad (2)$$

This forces the probability p_{same} to be zero, meaning that we can never unambiguously determine a set of transformations to be identical, unless we have additional information about the transforms, for example, that they belong to a certain given set of transforms.

In contrast, it is possible to obtain the unambiguous knowledge that the transformations are different. To treat the problem in its full generality, with the transforms to be compared appearing in an arbitrary network as described above, is difficult. We will limit the discussion to the case when the transforms appear in parallel, as shown for two transforms U and V in

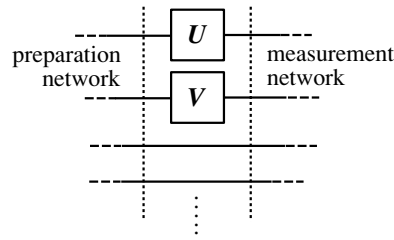


Figure 1. A general network for comparing two unitary transforms U and V .

figure 1. An input state is sent through a preparation network, then acted upon by U and V as shown, and finally measured. We have to be able to associate certain outcomes unambiguously with the transforms being different.

We will first concentrate on comparing two single-qubit transformations. These represent the simplest elements of quantum networks, and the ideas involved are demonstrated in a clear way. Here we present different comparison strategies, one using only a single qubit as a test state, and another strategy involving two qubits in a singlet state. These are shown to succeed more often than a third strategy using two non-entangled qubits. The strategies will then be generalized to comparison of more than two transforms, and to transforms of higher dimension. We end with conclusions.

2. Comparing two unitary-qubit transformations

We will start with comparing two unknown single-qubit transformations U and V . A general way of parametrizing 2×2 unitary transforms is to write them as $SU(2)$ -transforms with a global phase,

$$U = e^{i\varphi_u} \begin{pmatrix} a_u & -b_u \\ b_u^* & a_u^* \end{pmatrix} \quad V = e^{i\varphi_v} \begin{pmatrix} a_v & -b_v \\ b_v^* & a_v^* \end{pmatrix} \quad (3)$$

with $|a_{u(v)}|^2 + |b_{u(v)}|^2 = 1$. We see that four real parameters suffice to determine each transform. The global phase will matter whenever a unitary transform may or may not be applied to a physical system, such as in an interferometric setup, or in a controlled- U gate. For this reason, we cannot regard transforms which have different global phases, but otherwise are identical, as equivalent. Surprisingly enough, it is found that a single-particle test state is enough to test, in a single run, for differences in three of these four degrees of freedom. We will also consider the effect of entanglement using a two-particle singlet test state, contrasting this method to a two-particle non-entangled strategy.

2.1. Single-qubit strategy

Assume that, apart from two single copies of U and V , only one test qubit is available. Let us think about photons for the moment, with the two-qubit basis states being horizontal and vertical polarization, although the discussion applies to any two-level system. A strong reason for considering what may be done with only one test particle is that any unitary transform, acting on one photon, may be achieved with linear optical elements. This means that it will be possible to realize in an experiment any single-qubit test strategy.

We will start by considering one intuitive example strategy, then generalizing the argument to obtain an optimal single-particle comparison strategy. As shown in figure 2, we may for

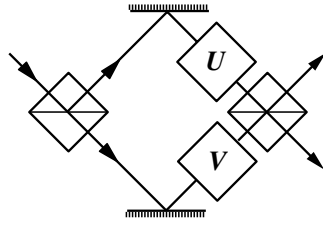


Figure 2. A network for comparing U and V with a single test particle. U and V are acting in each arm of an interferometer.

example split the photon with a beam splitter, not necessarily 50/50, let U and V act in one path each, recombine the two paths with another beam splitter, not necessarily identical to the first, and detect in which path and with what polarization the photon exits. Phase shifts, not included in the figure, may be added at any point in the network. The first beam splitter can be thought of merely as a state preparation device. A general single-photon test state is

$$|\psi_{\text{in}}\rangle = c \begin{pmatrix} 1 \\ 0 \end{pmatrix}_u + d \begin{pmatrix} 0 \\ 1 \end{pmatrix}_u + e \begin{pmatrix} 1 \\ 0 \end{pmatrix}_v + f \begin{pmatrix} 0 \\ 1 \end{pmatrix}_v \quad (4)$$

where $\begin{pmatrix} 1 \\ 0 \end{pmatrix}_{u(v)}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}_{u(v)}$ are the basis states of the photon in the paths going through U and V , and $|c|^2 + |d|^2 + |e|^2 + |f|^2 = 1$. Any state of this form can be prepared by a linear state preparation network built with polarizing and non-polarizing beam splitters, phase shifts and polarization rotations. After U and V acting in the paths, the state is given by

$$\begin{aligned} |\psi_{\text{out}}\rangle &= U \left[c \begin{pmatrix} 1 \\ 0 \end{pmatrix}_u + d \begin{pmatrix} 0 \\ 1 \end{pmatrix}_u \right] + V \left[e \begin{pmatrix} 1 \\ 0 \end{pmatrix}_v + f \begin{pmatrix} 0 \\ 1 \end{pmatrix}_v \right] \\ &= e^{i\varphi_u} \begin{pmatrix} a_u c - b_u d \\ b_u^* c + a_u^* d \end{pmatrix}_u + e^{i\varphi_v} \begin{pmatrix} a_v e - b_v f \\ b_v^* e + a_v^* f \end{pmatrix}_v. \end{aligned} \quad (5)$$

To get a unique signal of dissimilarity, we have to project on measurement states which are orthogonal to the output state obtained when $U = V$, i.e., to the state

$$|\psi_{\text{out}, U=V}\rangle = e^{i\varphi} \begin{pmatrix} ac - bd \\ b^*c + a^*d \end{pmatrix}_u + e^{i\varphi} \begin{pmatrix} ae - bf \\ b^*e + a^*f \end{pmatrix}_v. \quad (6)$$

The measurement is, in general, described by a probability operator measure (POM) strategy, but, in this case, only a simple projective measurement using a beam splitter (and phase shifts) will be required. We can choose for instance $d = f = 0$ and $c = e = 1/\sqrt{2}$, so that

$$|\psi_{\text{in}}\rangle = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}_u + \begin{pmatrix} 1 \\ 0 \end{pmatrix}_v \right] \quad (7)$$

representing a horizontally polarized photon, split equally between the two paths. When $U = V$, the output state is

$$|\psi_{\text{out}, U=V}\rangle = \frac{e^{i\varphi}}{\sqrt{2}} \left[\begin{pmatrix} a \\ b^* \end{pmatrix}_u + \begin{pmatrix} a \\ b^* \end{pmatrix}_v \right]. \quad (8)$$

This will always be orthogonal to the states

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}_u - \begin{pmatrix} 1 \\ 0 \end{pmatrix}_v \right] \quad |\psi_2\rangle = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}_u - \begin{pmatrix} 0 \\ 1 \end{pmatrix}_v \right]. \quad (9)$$

Whenever the photon is found exiting in either of these two states, U and V must have been different. In practice, the detection can be effected by recombining the two paths by a second non-polarizing beam splitter with the matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{10}$$

written in the (u, v) basis, meaning that if $U = V$, the output state is

$$|\psi_{\text{out}, U=V}\rangle = e^{i\varphi} \begin{pmatrix} a \\ b^* \end{pmatrix}_u \tag{11}$$

Whenever the photon is exiting in the v path, U and V must have been different. Note that even if U and V are not identical, the photon may be found in path u . We can calculate the success rate for detecting the two transformations as different. The result reads

$$P_{\text{diff}} = \langle \psi_{\text{out}} | \psi_1 \rangle \langle \psi_1 | \psi_{\text{out}} \rangle + \langle \psi_{\text{out}} | \psi_2 \rangle \langle \psi_2 | \psi_{\text{out}} \rangle \\ = \frac{1}{4} (|e^{i\varphi_u} a_u - e^{i\varphi_v} a_v|^2 + |e^{i\varphi_u} b_u - e^{i\varphi_v} b_v|^2). \tag{12}$$

If $U = V$, meaning $a_u = a_v, b_u = b_v$ and $\varphi_u = \varphi_v$, P_{diff} is equal to zero as required for an unambiguous result. If $U \neq V$, they will never be detected as different. If we want to obtain an overall measure for the detection efficiency, P_{diff} should be averaged over all possible transforms U and V :

$$\bar{P}_{\text{diff}} = \int dU \int dV P_{\text{diff}} \mu(U) \mu(V) = \frac{1}{2}. \tag{13}$$

Here, μ has to be taken as the correct group-invariant measure, given for $SU(2)$ in [14]. For $SU(2)$ with an additional overall phase φ , the correct measure is given by

$$dU \mu(U) = d\theta d\alpha d\beta d\varphi \frac{\cos 2\theta}{8\pi^3} \tag{14}$$

where $a = \cos \theta e^{i\alpha}, b = \sin \theta e^{i\beta}, 0 < \theta < \pi/2, 0 < \alpha < 2\pi, 0 < \beta < 2\pi, 0 < \varphi < 2\pi$ and similarly for $dV \mu(V)$. This result, $\bar{P}_{\text{diff}} = 1/2$, is due to the fact that the dimension of the subspace spanned by the states (9) is two out of a total of four dimensions. Integrating over all possible U and V corresponds to integrating over all possible output states (5) in the four-dimensional Hilbert space. Projecting onto two of these four dimensions obviously restricts the integration to these two dimensions, yielding $\bar{P}_{\text{diff}} = 2/4 = 1/2$.

The result obtained is not dependent on the polarization of the state $|\psi_{\text{in}}\rangle$ in equation (7). Any other basis is equally suited. In general, what we have to find is measurement states

$$|\psi_{\text{diff}}\rangle = \tilde{c} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_u + \tilde{d} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_u + \tilde{e} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_v + \tilde{f} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_v \tag{15}$$

orthogonal to the state $|\psi_{\text{out}, U=V}\rangle$ in equation (6). The fact that the orthogonality has to hold for all possible a, b and φ leads to the condition

$$\tilde{c}^* c + \tilde{e}^* e = \tilde{d}^* d + \tilde{f}^* f = \tilde{c}^* d + \tilde{e}^* f = \tilde{d}^* c + \tilde{f}^* e = 0 \tag{16}$$

giving

$$ed = cf \quad \text{and} \quad \tilde{e}\tilde{d} = \tilde{c}\tilde{f}. \tag{17}$$

This means that the polarization in the two paths has to be the same, but any polarization will do. The measurement states have to be adjusted according to (16) to match the polarization of the test state. The probability to detect a difference will be optimal when the probabilities for the photon to be in each path are equal, both equal to $1/2$. Thus, our intuitive first example of

strategy with the test state (7) proved to be one of the optimal one-photon test strategies with a test state of the form (4).

Let us note that the single-photon strategy fails to detect a difference in U and V whenever $P_{\text{diff}} = 0$ without the transforms being identical. If we write $a = |a|e^{i\alpha}$ and $b = |b|e^{i\beta}$, with $|b|^2 = 1 - |a|^2$, the strategy fails to distinguish the transforms characterized by $(|a|, \alpha, \beta, \varphi)$ and $(|a|, \alpha + \delta, \beta + \delta, \varphi - \delta)$, for any δ . Therefore, it effectively detects differences in three parameters out of the four necessary to parametrize a general single-qubit unitary transformation. It is of course possible to apply a mixed strategy, using two or more different polarizations for the test state, for example, a choice of two different polarizations with probability 1/2 each. One will then have a nonzero probability of detecting any kind of difference in U and V , even if the overall difference detection probability does not increase.

It is possible to further generalize the argument to any strategy where at most one particle passes through U or V , but allowing for example entanglement with an environment. The general input state then reads

$$|\psi_{\text{in}}\rangle = c|+\rangle_u|0\rangle_v|\phi_c\rangle_E + d|-\rangle_u|0\rangle_v|\phi_d\rangle_E + e|0\rangle_u|+\rangle_v|\phi_e\rangle_E + f|0\rangle_u|-\rangle_v|\phi_f\rangle_E + g|0\rangle_u|0\rangle_v|\phi_g\rangle_E \quad (18)$$

where $|+\rangle_{u,v} \equiv \binom{1}{0}_{u,v}$ and $|-\rangle_{u,v} \equiv \binom{0}{1}_{u,v}$ refer to single photons in the u or v path. The vacuum state $|0\rangle_{u,v}$ denotes that no photons are present in the u or v path. The states $|\phi\rangle_E$ are arbitrary but fixed environment states, describing the state of any particles which U and V will not explicitly act on, and may be selected so as to optimize the strategy. The constants c, d, e, f and g , whose absolute values sum to 1, are also chosen to optimize the strategy.

The transforms U and V are then allowed to act in their respective paths, giving an output state $|\psi_{\text{out}}\rangle$ exactly as before. By measuring $|\psi_{\text{out}}\rangle$, we now have to judge whether U and V are different or not. Any generalized measurement strategy may be described as a projective measurement in a higher-dimensional Hilbert space [15]. A requirement for an unambiguous comparison strategy is therefore that there exists at least one measurement state

$$|\psi_{\text{diff}}\rangle = \tilde{c}|+\rangle_u|0\rangle_v|\tilde{\phi}_{\tilde{c}}\rangle_E + \tilde{d}|-\rangle_u|0\rangle_v|\tilde{\phi}_{\tilde{d}}\rangle_E + \tilde{e}|0\rangle_u|+\rangle_v|\tilde{\phi}_{\tilde{e}}\rangle_E + \tilde{f}|0\rangle_u|-\rangle_v|\tilde{\phi}_{\tilde{f}}\rangle_E + \tilde{g}|0\rangle_u|0\rangle_v|\tilde{\phi}_{\tilde{g}}\rangle_E \quad (19)$$

which is orthogonal to the output state $|\psi_{\text{out}, U=V}\rangle$ whenever U and V are identical. Here, $|\tilde{\phi}\rangle_E$ denote environment states, and $\tilde{c}, \tilde{d}, \tilde{e}, \tilde{f}$ and \tilde{g} are constants. There may exist several linearly independent measurement states, as in the explicit example above.

As for the vacuum part of the general input state (18), the fact that the orthogonality condition has to hold for all $\varphi_u = \varphi_v$ leads to the requirement

$$\tilde{g}^* g \langle \tilde{\phi}_{\tilde{g}} | \phi_g \rangle = 0 \quad (20)$$

and although it is possible to have $g \neq 0$, after a quick consideration one realizes that $g = 0$ is optimal: due to condition (20), a nonzero g will not contribute to the success probability, but can only decrease it, since a nonzero g forces the absolute values of c, d, e and f to be smaller.

To investigate whether entanglement with the environment may be useful or not, let us not embark on any lengthy algebraic calculation, but instead use an intuitive argument. In the general input state (18), if $|\phi_c\rangle_E \neq |\phi_e\rangle_E$ or $|\phi_d\rangle_E \neq |\phi_f\rangle_E$, then it is possible to obtain which-path information, that is, knowledge about whether the single photon passed through U or V , by a measurement on the environment. This inevitably degrades the success rate of the comparison, which relies on our inability to determine whether the photon passed through U or V in the same way which-path information degrades the visibility in an interferometer. We are thus led to $|\phi_c\rangle_E = |\phi_e\rangle_E$ and $|\phi_d\rangle_E = |\phi_f\rangle_E$. On the other hand,

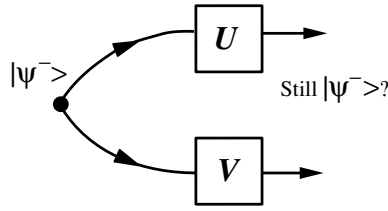


Figure 3. The transforms U and V are applied to a singlet test state. If the output state is not the singlet state, then U and V must have been different.

if $|\phi_c\rangle_E = |\phi_e\rangle_E \neq |\phi_d\rangle_E = |\phi_f\rangle_E$, this may be used to realize a mixed strategy employing either the test polarization $|+\rangle$ or $|-\rangle$ with desired probabilities. If $|\phi_c\rangle \perp |\phi_d\rangle$, the selection is made by a detection of the environment state in the $\{|\phi_c\rangle_E, |\phi_d\rangle_E\}$ basis. This will not affect the comparison success probability, as it is independent of the polarization of the test state.

In applications such as dense coding [16], the entanglement may be used to enhance the distinguishability of a set of unitary transforms. This is partly due to the enlarged Hilbert space arising from having two qubits instead of one. Here we have effectively enlarged the dimension of the available Hilbert space from two to four by introducing an extra degree of freedom—a choice of two paths—to a single qubit (photon), without the explicit use of entanglement. As another option, entanglement between two photons, one passing through U , the other through V , may enhance the comparison success rate, not always, but in certain cases. This will be considered next.

2.2. Two-qubit singlet strategy

Let us consider a strategy which involves the use of a singlet state. Experimentally, a singlet state of two photons may be prepared using parametric down-conversion. The test input state reads

$$|\psi_{\text{in}}\rangle = |\psi^-\rangle = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}_u \begin{pmatrix} 0 \\ 1 \end{pmatrix}_v - \begin{pmatrix} 0 \\ 1 \end{pmatrix}_u \begin{pmatrix} 1 \\ 0 \end{pmatrix}_v \right]. \tag{21}$$

The singlet has the property that it remains invariant under the action of any two-particle transformation of the form $U_u \otimes U_v$. The output state after passing through the transform $U_u \otimes V_v$ as in figure 3 reads

$$|\psi_{\text{out}}\rangle = e^{i(\varphi_u + \varphi_v)} [\text{Re}(a_u a_v^* + b_u b_v^*) |\psi_{\text{in}}\rangle + i \text{Im}(a_u a_v^* + b_u b_v^*) |\psi^+\rangle + \text{Re}(b_u a_v - a_u b_v) |\Phi^+\rangle + i \text{Im}(b_u a_v - a_u b_v) |\Phi^-\rangle] \tag{22}$$

where

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}_u \begin{pmatrix} 1 \\ 0 \end{pmatrix}_v \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix}_u \begin{pmatrix} 0 \\ 1 \end{pmatrix}_v \right] \tag{23}$$

and

$$|\psi^\pm\rangle = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}_u \begin{pmatrix} 0 \\ 1 \end{pmatrix}_v \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix}_u \begin{pmatrix} 1 \\ 0 \end{pmatrix}_v \right] \tag{24}$$

are the Bell states. If $U = V$, then the output state is

$$|\psi_{\text{out}}\rangle = e^{2i\varphi} |\psi_{\text{in}}\rangle. \tag{25}$$

If the output state is found in any Bell state other than the singlet, then this indicates that the transformations are different. This requires a projection onto the antisymmetric versus the

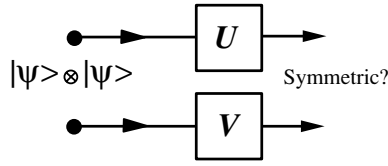


Figure 4. The transforms U and V are applied to two non-entangled test qubits. If their output state is found to be antisymmetric, then U and V must have been different.

symmetric subspace, and may be implemented with a non-polarizing beam splitter [17]. The overall success probability reads

$$P_{\text{diff}} = 1 - [\text{Re}(a_u a_v^* + b_u b_v^*)]^2. \quad (26)$$

The averaged success rate, in analogy with equation (13), is found to be $3/4$. This is because the measurement space, spanned by the three Bell states other than the singlet, is now three-dimensional. We should note that it is also possible to start with any $|\psi_{\text{in}}\rangle$ in the subspace $\hat{1} - |\psi^-\rangle\langle\psi^-|$ orthogonal to the singlet state, whereby a detection of the output state in $|\psi^-\rangle$ indicates that the transforms were different. In this case, however, the comparison success rate will be only $1/4$, since the measurement space now has only one dimension out of four.

The singlet strategy can discriminate between all possible $SU(2)$ transformations, but cannot detect global phase differences $\varphi_u - \varphi_v$. This means that, like the single-photon strategy, it can detect differences in three of the four parameters. Two runs of the single-photon strategy with average success rate $1/2$ each will have the same overall success rate, $3/4$, as one run of the singlet strategy. Thus it would seem as if the use of entanglement carries limited advantage. However, in the strictest sense of the requirement ‘only one use of U and V are allowed’, two runs of the single-photon strategy would not be permitted, even if the average number of photons passing through U and V is one each. Also, the singlet strategy is more efficient when it comes to detecting small differences in the $SU(2)$ part of U and V . For a difference $\Delta\theta = \theta_u - \theta_v$, the single-photon and singlet strategies have difference detection probabilities of $\sin \Delta\theta$ and $\sin 2\Delta\theta$, respectively, meaning that for a small difference $\Delta\theta$, the singlet strategy is a factor of four better than the one-photon strategy, or a factor of two better than two single-photon runs. On the other hand, the single-photon interferometric scheme is capable of detecting overall phase differences, where the singlet strategy fails. The choice of strategy depends on the type of difference to be detected. As already noted above, the single-photon strategy introduces extra degrees of freedom to a single photon in allowing two different paths, so that the dimensionality of the test space for the single-photon strategy (four) equals that of the singlet strategy (four).

2.3. A non-entangled two-particle strategy

We may also compare the singlet strategy to a non-entangled two-particle strategy based on state comparison [12]. Starting with two particles in the same state $|\Psi\rangle$, apply U to one of them and V to the other, as in figure 4. There exists no unambiguous test telling us whether the particles are still in the same state, but if they are found to be different, then we can infer that U and V must have been different. The probability for this strategy to succeed in detecting a difference in $U|\Psi\rangle$ and $V|\Psi\rangle$ is [12]

$$P_{\text{diff}} = \langle\Psi|\langle\Psi|\psi^-\rangle\langle\psi^-|\Psi\rangle|\Psi\rangle = \frac{1}{2}(1 - |\langle\Psi|U^\dagger V|\Psi\rangle|^2). \quad (27)$$

Choosing $|\Psi\rangle = \binom{1}{0}$ (the integrated overall success probability cannot depend on the state chosen if the transforms are completely unknown), and using the facts that $\text{Re}(a_u a_v^* + b_u b_v^*) = \text{Re}(a_u a_v^* + b_u^* b_v) \leq |a_u a_v^* + b_u b_v^*| = |\langle\Psi|U^\dagger V|\Psi\rangle|$ and $\text{Re}(a_u a_v^* + b_u b_v^*) \leq 1$, this can be seen to be less than or equal to the success probability of the singlet strategy. The difference in comparison with the singlet strategy is a manifestation of gain due to entanglement. The overall averaged success probability for one run of the non-entangled two-photon strategy is $1/4$. This may be realized by integrating equation (27) directly, or by noting that detecting $U|\Psi\rangle$ and $V|\Psi\rangle$ as different corresponds to finding them in an antisymmetric state, in other words, in $|\psi^-\rangle$, which is one of four orthogonal basis states.

An advantage of the single-photon and non-entangled strategies is that they are easily generalized to comparison of many transforms and transforms of higher dimension, as will be seen below.

3. Comparison of many unitary-qubit transforms

It is straightforward to generalize the single-photon comparison strategy to the case when N 2×2 unitary transformations are to be compared. Denote the transforms by $U_i, i = 1, \dots, N$, where

$$U_i = e^{i\varphi_i} \begin{pmatrix} a_i & -b_i \\ b_i^* & a_i^* \end{pmatrix}. \tag{28}$$

We want to find a strategy which tells us when all U_i are not identical, that is, when at least one U_i differs from the rest. In analogy with the comparison between two 2×2 unitary transforms, use the single-photon test state

$$|\psi_{\text{in}}\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^N \binom{1}{0}_i \tag{29}$$

where $\binom{1}{0}_i$ and $\binom{0}{1}_i$ are the basis states (e.g. polarization) of a particle in the path going through U_i . This test state may be obtained using a non-polarizing beam splitter, which is a linear device splitting an incident particle into N different paths with equal probability $1/N$ each, independent of the polarization [18]. As when comparing two transforms, the same freedom in choosing the polarization applies here; as long as the polarization is the same in all paths, any polarization may be used. After action with U_i in the different paths, the output state reads

$$|\psi_{\text{out}}\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^N e^{i\varphi_i} \binom{a_i}{b_i^*}_i. \tag{30}$$

If all U_i are identical, meaning $a_i = a, b_i = b$ and $\varphi_i = \varphi$ for all i , then

$$\begin{aligned} |\psi_{\text{out,id}}\rangle &= \frac{e^{i\varphi}}{\sqrt{N}} \sum_{i=1}^N \binom{a}{b^*}_i = e^{i\varphi} \left[a \frac{1}{\sqrt{N}} \sum_{i=1}^N \binom{1}{0}_i + b^* \frac{1}{\sqrt{N}} \sum_{i=1}^N \binom{0}{1}_i \right] \\ &\equiv e^{i\varphi} (a|\Psi_{\text{id}}^1\rangle + b^*|\Psi_{\text{id}}^2\rangle). \end{aligned} \tag{31}$$

This state is orthogonal to states of the form

$$|\Psi_{ij}^{1-}\rangle = \frac{1}{\sqrt{2}} \left[\binom{1}{0}_i - \binom{1}{0}_j \right] \quad |\Psi_{ij}^{2-}\rangle = \frac{1}{\sqrt{2}} \left[\binom{0}{1}_i - \binom{0}{1}_j \right] \quad i \neq j. \tag{32}$$

These states form an overcomplete basis for the subspace orthogonal to the one spanned by states of the form (31). Whenever the photon is detected in this orthogonal subspace, this

shows unambiguously that at least one of the U_i was not identical to the others. The detection can be effected by combining the N paths with a second non-polarizing $2N$ -port beam splitter, similar to the one used for the test state preparation. The state $|\psi_{\text{out,id}}\rangle$ will then be mapped onto $e^{i\varphi} \binom{a}{b}_1$, and whenever the photon is exiting in a path $i \neq 1$, we know that the U_i were not all the same.

The success probability to detect a difference reads

$$\begin{aligned} P_{\text{diff},N} &= \frac{2}{N} \sum_{i,j=1;i>j}^N \langle \psi_{\text{out}} | \Psi_{ij}^{1-} \rangle \langle \Psi_{ij}^{1-} | \psi_{\text{out}} \rangle + \langle \psi_{\text{out}} | \Psi_{ij}^{2-} \rangle \langle \Psi_{ij}^{2-} | \psi_{\text{out}} \rangle \\ &= \frac{1}{N^2} \sum_{i,j=1;i>j}^N (|e^{i\varphi_i} a_i - e^{i\varphi_j} a_j|^2 + |e^{i\varphi_i} b_i - e^{i\varphi_j} b_j|^2). \end{aligned} \quad (33)$$

To obtain the overall success probability to detect differences in N transforms, we should integrate $P_{\text{diff},N}$ over all possible transforms U_i . Alternatively, we may use

$$P_{\text{diff},N} = 1 - \langle \psi_{\text{out}} | \Psi_{\text{id}}^1 \rangle \langle \Psi_{\text{id}}^1 | \psi_{\text{out}} \rangle - \langle \psi_{\text{out}} | \Psi_{\text{id}}^2 \rangle \langle \Psi_{\text{id}}^2 | \psi_{\text{out}} \rangle. \quad (34)$$

Integrating this over U_i , we obtain $\bar{P}_{\text{diff},N} = 1 - 1/N$. Again, this is consistent with the fact that the dimension of the subspace spanned by the states in equation (32) is $2N - 2$, the dimension of the total Hilbert space being $2N$, so that $\bar{P}_{\text{diff},N} = (2N - 2)/2N = 1 - 1/N$. As the number of transforms grows, it is, loosely speaking, more likely that at least one of them differs from the rest, and our probability to detect a difference approaches unity. Of course, for any number of transforms selected at random, the chance that they are all exactly equal is zero; in this sense, the probability for randomly selected transforms to be different is one. But when the number of transforms increase, our probability of unambiguously detecting a difference also approaches unity.

In analogy with the non-entangled two-particle comparison strategy for two transforms in section 2.3, it is also possible to compare many unitary transforms by applying each transform to a copy of a state $|\Psi\rangle$, and then comparing the states. Whenever all the transforms are identical, all the states will also be identical, and the total state will be symmetric. Thus, whenever the joint state of all the test particles is found outside the totally symmetric subspace, the transforms cannot all have been identical. (The particles may be distinguished through their spatial locations, otherwise we could not apply one unitary transform to each specific particle.) The overall comparison success rate \bar{P}_{diff} will be equal to $1 - D_s/2^N$, where 2 is the dimension of each unitary transform, N is the number of transforms to be compared and D_s is the dimension of the totally symmetric subspace for N two-dimensional systems.

For example, for comparing three 2×2 transforms, one may construct the three-particle states

$$|\psi_{ij}^0\rangle = \frac{1}{\sqrt{2}}(|01\rangle_{ij} - |10\rangle_{ij})|0\rangle_k \quad \text{and} \quad |\psi_{ij}^1\rangle = \frac{1}{\sqrt{2}}(|01\rangle_{ij} - |10\rangle_{ij})|1\rangle_k \quad (35)$$

where i, j and k may be any permutation of 1, 2 and 3, referring to the three particles, and $|0\rangle, |1\rangle$ are the single-particle basis states. The projector onto the non-totally symmetric subspace may then be formed as

$$\hat{1} - \hat{P}_{\text{symm}} = \frac{1}{3} \sum_{i \neq j} (|\psi_{ij}^0\rangle \langle \psi_{ij}^0| + |\psi_{ij}^1\rangle \langle \psi_{ij}^1|). \quad (36)$$

It follows that this method is related to pairwise comparison of the particle states.

4. Comparison of unitary transforms of higher dimension

It is also possible to consider comparison of unitary transforms with a dimension higher than 2×2 . Below we will briefly suggest strategies for this. Physically, a 4×4 transform may describe a two-qubit operation, or a single-particle operation on a single physical system with four dimensions. A ‘mathematical’ comparison strategy derived for comparing 4×4 unitary transforms would work in both cases, but the physical implementations are different.

As before, we have to act with the transforms on a test state, which may be a multi-particle state, selecting the test state and the way the transforms are used so that certain outcomes indicate unambiguously that the transforms were different. It is straightforward to generalize the single-particle strategy used above; we will sketch the extension here. Let us consider the case of two $M \times M$ unitary matrices U and V , with matrix elements a_{ij} and b_{ij} . Denote the M basis states in the u and v paths by $|i\rangle_{u,v}$, where $i = 1, \dots, M$. Using the test state

$$|\psi_{\text{in}}\rangle = \frac{1}{\sqrt{2}}(|1\rangle_u + |1\rangle_v) \quad (37)$$

we may compare the first columns of U and V . The test state

$$|\psi_{\text{in}}\rangle = \frac{1}{\sqrt{2}}(|i\rangle_u + |i\rangle_v) \quad (38)$$

compares the i th columns of U and V . If a general comparison of all columns is desired, then we may employ a mixed strategy, e.g. selecting the M states (38) with probability $1/M$ each. The success probability to detect a difference in U and V for this mixed strategy is

$$P_{\text{diff}} = \frac{1}{4M} \sum_{i,j=1}^M |a_{ij} - b_{ij}|^2. \quad (39)$$

If U and V are single-particle transforms, then this is still a true single-particle comparison strategy, meaning that it is possible to implement for example with linear optics. Otherwise, if U and V are multi-particle operations, the test state is a multi-particle state, the basis states $|i\rangle$ being multi-particle states. It is a straightforward mathematical task to generalize this type of comparison strategy to the case of more than two $M \times M$ transforms. The averaged comparison success probability will be $1 - 1/N$, where N is the number of transforms.

The singlet strategy in section 2.2 employed the concept of an invariant subspace, as the singlet is invariant under $U \otimes U$. When comparing N N -dimensional unitary transforms, it is possible to generalize this idea. A totally antisymmetric state of N N -dimensional particles (a Slater determinant) is invariant if the same unitary transform is applied to all N particles. The detection of the system in any other state than the totally antisymmetric state after application of one transform to each particle therefore means that the transforms cannot all have been identical. It is here understood that the Slater determinant refers to the internal degree of freedom of the N particles, which may be distinguished because of different spatial location. Otherwise we could not apply one specific transform to each particle. The dimension of the total internal Hilbert space is therefore N^N , and, since there is only one totally antisymmetric state, the averaged comparison success probability is $1 - 1/N^N$. This may be compared with the averaged success probability of the single-particle strategy, $1 - 1/N$. A non-entangled strategy based on state comparison succeeds with a probability $1 - D_s/N^N$, where D_s is the dimension of the totally symmetric subspace.

If the number of transforms to be compared is not equal to their dimensionality, then we still have to look for the lowest dimensional invariant subspaces. We may choose a test state lying in the lowest dimensional invariant subspace available, apply the transforms to

be compared, and then check whether the output state remains in the same subspace or not. Strictly speaking the subspace does not even have to be invariant, it will suffice if we are able to unambiguously predict in which subspace H_{id} the output state will lie if the transforms are identical. An output state in the space complementary to H_{id} will then indicate a difference in the transforms. The comparison of two two-qubit transformations, for instance, would require us to work with the following four-particle states:

$$\begin{aligned}
 |\psi_1\rangle &= \frac{1}{\sqrt{2}}(|1\rangle_1|0\rangle_2|0\rangle_3|1\rangle_4 - |0\rangle_1|1\rangle_2|1\rangle_3|0\rangle_4) \\
 |\psi_2\rangle &= \frac{1}{\sqrt{2}}(|1\rangle_1|1\rangle_2|0\rangle_3|0\rangle_4 - |0\rangle_1|0\rangle_2|1\rangle_3|1\rangle_4) \\
 |\psi_3\rangle &= \frac{1}{\sqrt{2}}|1\rangle_1|1\rangle_3(|0\rangle_2|1\rangle_4 - |1\rangle_2|0\rangle_4) \\
 |\psi_4\rangle &= \frac{1}{\sqrt{2}}|0\rangle_1|0\rangle_3(|0\rangle_2|1\rangle_4 - |1\rangle_2|0\rangle_4) \\
 |\psi_5\rangle &= \frac{1}{\sqrt{2}}|0\rangle_2|0\rangle_4(|0\rangle_1|1\rangle_3 - |1\rangle_1|0\rangle_3) \\
 |\psi_6\rangle &= \frac{1}{\sqrt{2}}|1\rangle_2|1\rangle_4(|0\rangle_1|1\rangle_3 - |1\rangle_1|0\rangle_3).
 \end{aligned} \tag{40}$$

These states span the non-totally symmetric subspace ($|0\rangle$ and $|1\rangle$ are the qubit basis states). Its complementary subspace is the symmetric subspace, which is ten-dimensional. A test state lying either in the subspace spanned by the states (40) or in the totally symmetric subspace will remain in the same subspace whenever the two transforms are identical. A detection of the output state in the complementary subspace therefore unambiguously implies that the transforms were different. Starting in the lower dimensional non-totally symmetric subspace, however, will yield a higher comparison success probability, $D_s/4^2 = 10/16$, as opposed to $1 - 10/16$ when starting in the ten-dimensional symmetric subspace.

The non-entangled strategy based on state comparison makes use of the totally symmetric subspace as a starting space. We may use it for comparing any number of unitary transforms of any dimension, by acting with the transforms to be compared upon copies of a test state, and then comparing all the test states. Although an entangled strategy may perform better, because it can use an invariant subspace with a lower dimension, a non-entangled strategy may be easier to implement experimentally. The single-particle strategy is clearly the easiest to realize. Its success probability also scales favourably as $1 - 1/N$ with the number of transforms.

5. Partially known transforms, decoherence and imperfections

If we have additional information about the transforms, then we may use this knowledge in selecting the test state and the way in which to use the (single) copies of the transforms available, and in how to detect differences and similarities from the output state. This is possible whatever strategy chosen, using a single particle or a many particle test state, for unambiguous and minimum-error strategies, or for any other type of comparison strategy. The same freedom to select for example entangled test states applies.

It will always be possible to use the universal comparison strategies considered above, but it may be possible to improve on the success probability P_{diff} , and also to obtain an unambiguous answer that the transformations are identical. This is the case if the two transforms are known to be members of a given set, e.g. $\{U_1, U_2\}$. There are then four possibilities for the output state, depending on whether $U = V = U_1$, $U = V = U_2$, $U_1 = U \neq V = U_2$, or $U_2 = U \neq V = U_1$, and we have to distinguish the two former ones from the two latter ones. In other words, we have to distinguish the subspace spanned by the two output states when $U = V$ from the subspace spanned by the output states when $U \neq V$. In

general, these two subspaces are nonorthogonal, and cannot be perfectly discriminated. But in analogy with state discrimination, it will be possible to unambiguously distinguish these two possibilities, allowing for inconclusive results. It will be possible to obtain an unambiguous answer not only that U and V are different, but also that they are identical. We could also choose to make a minimum-error measurement.

If we are given the information that U and V are both controlled-NOT gates, but not in which basis, then, if they have the same basis, UV will be the identity operator. This implies that if UV applied to a state changes the state, the bases of the two controlled-NOT gates U and V must have been different, and this can be used for comparison. This same argument may be useful when any cyclic transformations are to be compared.

Throughout the analysis in this paper, we have taken for granted that U and V are perfect unitary transformations, that the test states are perfect pure states, that no decoherence occurs, that detectors are perfect, etc. In real experiments, several of these conditions may not be met. Accounting for decoherence and imperfections is beyond the scope of this paper, but let us note that completely unambiguous answers may not be possible in the general case. This is because unambiguous comparison of, or discrimination between, density matrices is not always possible. If the test state after the comparison procedure is not in a pure state, we may not be able to obtain an absolutely certain answer. The minimum-error comparison strategies will still be possible to realize, giving an answer with the least possible error. Strategies related to unambiguous comparison may still be useful, for example, performing the same unambiguous strategy when decoherence and imperfections are present may give a lower error probability for the answer ‘different’ than the overall minimum-error strategy. It would be interesting to investigate situations like this.

6. Conclusions

In this paper, we have considered the situation where one is interested in whether two or more unitary transforms are the same or different. No knowledge about the individual transforms is desired, only whether there is a difference or not. It is indeed possible to *compare* the transforms without first estimating them. A ‘trial’ copy of a transform may also be compared to a master copy. This can be used to test for defects in the ‘trial’ copy, and may, in this sense, be used as a way of transformation estimation. We have considered unambiguous comparison strategies, which allows us to make confident statements about the transforms. When the transforms are completely unknown, it is not possible to obtain an unambiguous result that the transforms are identical, but it is possible to detect them as different.

We have treated the case of comparing two 2×2 unitary transforms in depth, and made some generalizations. Three types of unambiguous comparison strategies were derived: a single-particle strategy, a singlet strategy employing the concept of invariant subspaces, and a non-entangled two-particle strategy. Success probabilities are calculated for these strategies. Obvious extensions to the present work include a deeper treatment of transformation comparison when there is knowledge about the transforms, and to allow for decoherence and imperfections.

From an experimental point of view, it is harder to prepare single photons or photon pairs, than it is to use light pulses with many photons, for instance in the form of coherent states. We have considered the case when only one application of the unitary transformations is allowed. One immediately realizes that the setup for the single-photon strategy can be used with cw, or pulsed, laser light as well, as long as each photon is coherently split to pass through both U and V , and then coherently recombined again. The setup is similar to an interferometer. The intensity ratio between the photons detected in the u - and v -paths will provide an estimate of

the difference between the transforms. With growing pulse amplitude, this estimate can be made increasingly accurate.

The conceptual connection to state comparison is obvious [12]. One may also draw an analogy to the Deutsch-Jozsa algorithm for determining whether a function is balanced or constant with a single application of the function [19]. Here, the function values are not interesting, only whether the function is balanced or equal. Similarly, we have devised strategies for determining whether unitary transforms are different or not, without obtaining information about what the transforms actually are. The feature that collective information may be obtained without knowledge of the parts of a system is characteristic of many quantum algorithms versus their classical counterparts. It is connected to the possibility of choosing the measurement basis for a quantum system more freely than for a classical system.

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